MAXIMALLY HOMOGENEOUS PARA-CR MANIFOLDS OF SEMISIMPLE TYPE

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Abstract. An almost para-CR structure on a manifold M is given by a distribution $HM \subset TM$ together with a field $K \in \Gamma(\mathrm{End}(HM))$ of involutive endomorphisms of HM. If K satisfies an integrability condition, then (HM, K) is called a para-CR structure. The notion of maximally homogeneous para-CR structure of a semisimple type is given. A classification of such maximally homogeneous para-CR structures is given in terms of appropriate gradations of real semisimple Lie algebras.

Contents

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1. Introduction and notation

Let M be a $2n$-dimensional manifold. An *almost paracomplex structure* on M is a field of endomorphisms $K \in \mathrm{End}(TM)$ of the tangent bundle TM of M such that $K^2 = \mathrm{id}$. It is called an (almost) paracomplex structure in the *strong sense* if its ± 1-eigenspace distributions

$$T^{\pm}M = \{X \pm KX \mid X \in \Gamma(M, TM)\}$$

have the same rank (see e.g. [13], [9]). An almost paracomplex structure K is called a *paracomplex structure*, if it is *integrable*, i.e.

$$S(X,Y) = [X,Y] + [KX,KY] - K[X,KY] - K[KX,Y] = 0$$

for any vector fields X, $Y \in \Gamma(TM)$.
This is equivalent to say that the distributions $T^{\pm}M$ are involutive.

Recall that an *almost* CR-*structure* of codimension k on a $2n+k$-dimensional manifold M is a distribution $HM \subset TM$ of rank $2n$ together with a field of endomorphisms $J \in \mathrm{End}(HM)$ such that $J^2 = -\mathrm{id}$. An almost CR-structure is called CR-*structure*, if the $\pm i$-eigenspace subdistributions $H_{\pm}M$ of the complexified tangent bundle $T^{\mathbb{C}}M$ are involutive.
We define an almost para-CR structure in a similar way.

Definition 1.1. *An* almost para-CR structure *of codimension k on a $2n+k$-dimensional manifold M (in the weak sense) is a pair (HM, K), where $HM \subset TM$ is a rank $2n$ distribution and $K \in \mathrm{End}(HM)$ is a field of endomorphisms such that $K^2 = \mathrm{id}$ and $K \neq \pm\mathrm{id}$.*
An almost para-CR structure is said to be a para-CR structure*, if the eigenspace subdistributions $H_{\pm}M \subset HM$ are integrable or equivalently if the following integrability conditions hold:*

$$[KX,KY] + [X,Y] \in \Gamma(HM)\,, \tag{1}$$

$$S(X,Y) := [X,Y] + [KX,KY] - K([X,KY] + [KX,Y]) = 0 \tag{2}$$

for all X, $Y \in \Gamma(HM)$.

If the eigenspace distributions

$$H_{\pm}M = \{X \pm KX \mid X \in \Gamma(M, HM)\}$$

of an almost para-CR structure have the same rank, then (HM, K) is called an almost para-CR structure in the *strong sense*.
A straightforward computation shows that the integrability condition is equivalent to the involutivness of the distributions H_+M and H_-M.
A manifold M, endowed with an (almost) para-CR structure, is called an (*almost*) *para*-CR *manifold* .
Note that a direct product of (almost) para-CR manifolds is an (almost) para-CR manifold.
One can associate with a point $x \in M$ of a para-CR manifold (M, HM, K) a fundamental graded Lie algebra $\mathfrak{m}$. A para-CR structure is said to be *regular* if these Lie algebras $\mathfrak{m}_x$ do not depend on x. In this case, a para-CR

structure can be considered as a Tanaka structure (see [3] and section 4). A regular para-CR structure is called a structure of *semisimple type* if the full prolongation

$$\mathfrak{g} = \mathfrak{m}^{\infty} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots$$

of the associated non-positively graded Lie algebra $\mathfrak{g}^{-d} + \cdots + \mathfrak{g}^{-1} + \mathfrak{g}^0$ (which is an analogue of the generalized Levi form of a CR structure) is a semisimple Lie algebra. Such a para-CR structure defines a parabolic geometry and its group of automorphisms $\mathrm{Aut}(M, HM, K)$ is a Lie group of dimension $\leq \dim \mathfrak{g}$.

Recently in [16] P. Nurowski and G. A. J. Sparling consider the natural para-CR structure which arises on the 3-dimensional space M of solutions of a second order ordinary differential equation $y'' = Q(x, y, y')$. Using the Cartan method of prolongation, they construct the full prolongation $\mathcal{G} \to M$ of M with a $\mathfrak{sl}(3, \mathbb{R})$-valued Cartan connection and a quotient line bundle over M with a conformal metric of signature $(2, 2)$. This is a para-analogue of the Feffermann bundle of a CR-structure. They apply these bundles to the initial ODE and get interesting applications.

In [2] we proved that a para-CR structures of semisimple type on a simply connected manifold M with the automorphism group of maximal dimension $\dim \mathfrak{g}$ can be identified with a (real) flag manifold $M = G/P$ where G is the simply connected Lie group with the Lie algebra $\mathfrak{g}$ and P the parabolic subgroup generated by the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^d$. We gave a classification of maximally homogeneous para-CR structures of semisimple type such that the associated graded semisimple Lie algebra $\mathfrak{g}$ has depth $d = 2$. In the present paper we classify all maximally homogeneous para-CR structures of semisimple type in terms of graded real semisimple Lie algebras.

2. Graded Lie algebras associated with para-CR structures

2.1. Gradations of a Lie algebra. Recall that a *gradation* (more precisely a $\mathbb{Z}$-gradation) of *depth* k of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}^i = \mathfrak{g}^{-k} + \mathfrak{g}^{-k+1} + \cdots + \mathfrak{g}^0 + \cdots + \mathfrak{g}^j + \cdots \tag{3}$$

such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$, for any $i, j \in \mathbb{Z}$, and $\mathfrak{g}^{-k} \neq \{0\}$. Note that $\mathfrak{g}^0$ is a subalgebra of $\mathfrak{g}$ and each $\mathfrak{g}^i$ is a $\mathfrak{g}^0$-module.

We say that an element $x \in \mathfrak{g}^j$ has *degree* j and we write $d(x) = j$. The endomorphism δ of $\mathfrak{g}$ defined by

$$\delta_{|\mathfrak{g}_j} = j \cdot id$$

is a semisimple derivation of $\mathfrak{g}$ (with integer eigenvalues), whose eigenspaces determine the gradation. Conversely, any semisimple derivation δ of $\mathfrak{g}$ with integer eigenvalues defines a gradation where the grading space $\mathfrak{g}^j$ is the eigenspace of δ with eigenvalue j. If $\mathfrak{g}$ is a semisimple Lie algebra, then any

derivation δ is inner, i.e. there exists $d \in \mathfrak{g}$ such that $\delta = \mathrm{ad}_d$. The element $d \in \mathfrak{g}$ is called the *grading element*.

Definition 2.1. *A gradation $\mathfrak{g} = \sum \mathfrak{g}^i$ of a Lie algebra (or a graded Lie algebra $\mathfrak{g}$) is called*

(1) fundamental*, if the negative part $\mathfrak{m} = \sum_{i<0} \mathfrak{g}^i$ is generated by $\mathfrak{g}^{-1}$;*
(2) (almost) effective *or* transitive*, if the non-negative part*

$$\mathfrak{g}^{\geq 0} = \mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots$$

contains no non-trivial ideal of $\mathfrak{g}$;
(3) non-degenerate*, if*

$$X \in \mathfrak{g}^{-1}\,, \ [X, \mathfrak{g}^{-1}] = 0 \implies X = 0\,.$$

2.2. **Fundamental algebra associated with a distribution.** Let $\mathcal{H}$ be a distribution on a manifold M. We recall that to any point $x \in M$ it is possible to associate a Lie algebra $\mathfrak{m}(x)$ in the following way.
First of all, we consider a filtration of the Lie algebra $\mathcal{X}(M)$ of vector fields defined inductively by

$$\begin{aligned} \Gamma(\mathcal{H})_{-1} &= \Gamma(\mathcal{H})\,, \\ \Gamma(\mathcal{H})_{-i} &= \Gamma(\mathcal{H})_{-i+1} + [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})_{-i+1}] \text{ ,for } i > 1. \end{aligned}$$

Then evaluating vector fields at a point $x \in M$, we get a flag

$$T_xM \supset \mathcal{H}_{-d-1}(x) = \mathcal{H}_{-d}(x) \supsetneq \mathcal{H}_{-d+1}(x) \supset \cdots \supset \mathcal{H}_{-2}(x) \supset \mathcal{H}_{-1}(x) = \mathcal{H}_x$$

in T_xM, where

$$\mathcal{H}_{-i}(x) = \{X_{|_x} \mid X \in \Gamma(\mathcal{H})_{-i}\}.$$

Let us assume that $\mathcal{H}_{-d}(x) = T_xM$. The commutators of vector fields induce a structure of fundamental negatively graded Lie algebra on the associated graded space

$$\mathfrak{m}(x) = \mathrm{gr}(T_xM) = \mathfrak{m}^{-d}(x) + \mathfrak{m}^{-d+1}(x) + \cdots + \mathfrak{m}^{-1}(x)\,,$$

where $\mathfrak{m}^{-j}(x) = \mathcal{H}_{-j}(x)/\mathcal{H}_{-j+1}(x)$. Note that $\mathfrak{m}^{-1}(x) = \mathcal{H}_x$.
A distribution $\mathcal{H}$ is called a *regular distribution* of *depth d* and *type* $\mathfrak{m}$ if all graded Lie algebras $\mathfrak{m}(x)$ are isomorphic to a given graded fundamental Lie algebra

$$\mathfrak{m} = \mathfrak{m}^{-d} + \mathfrak{m}^{-d+1} + \cdots + \mathfrak{m}^{-1}\,.$$

In this case $\mathfrak{m}$ is called the *Lie algebra associated* with the distribution $\mathcal{H}$. A regular distribution $\mathcal{H}$ is called *non-degenerate* if the associated Lie algebra is non-degenerate.

2.3. **Para-CR algebras and regular para-CR structures.** We recall the following

Definition 2.2. *A pair* $(\mathfrak{m}, K_o)$*, where* $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ *is a negatively graded fundamental Lie algebra and* K_o *is an involutive endomorphism of* $\mathfrak{m}^{-1}$*, is called a* para-CR algebra *of* depth d*. If, moreover, the* ± 1*-eigenspaces* $\mathfrak{m}^{-1}_{\pm}$ *of* K_o *on* $\mathfrak{m}^{-1}$ *are commutative subalgebras of* $\mathfrak{m}$*, then* $(\mathfrak{m}, K_o)$ *is called an* integrable para-CR algebra*.*

Definition 2.3. *Let* $(\mathfrak{m}, K_o)$ *be a para-CR algebra of depth* d*. An almost para-CR structure* (HM, K) *on* M *is called* regular *of* type $(\mathfrak{m}, K_o)$ *and* depth d *if, for any* $x \in M$*, the pair* $(\mathfrak{m}(x), K_x)$ *is isomorphic to* $(\mathfrak{m}, K_o)$*. We say that the regular almost para-CR structure is non-degenerate if the graded algebra* $\mathfrak{m}$ *is non-degenerate.*

Note that a regular almost para-CR structure of type $(\mathfrak{m}, K_0)$ is integrable if and only if the Lie algebra $(\mathfrak{m}, K_0)$ is integrable.

3. Prolongations of graded Lie algebras

3.1. **Prolongations of negatively graded Lie algebras.** The *full prolongation* of a negatively graded fundamental Lie algebra $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ is defined as a maximal graded Lie algebra

$$\mathfrak{g}(\mathfrak{m}) = \mathfrak{g}^{-d}(\mathfrak{m}) + \cdots + \mathfrak{g}^{-1}(\mathfrak{m}) + \mathfrak{g}^{0}(\mathfrak{m}) + \mathfrak{g}^{1}(\mathfrak{m}) + \cdots$$

with the negative part

$$\mathfrak{g}^{-d}(\mathfrak{m}) + \cdots + \mathfrak{g}^{-1}(\mathfrak{m}) = \mathfrak{m}$$

such that the following transitivity condition holds:

$$\text{if } X \in \mathfrak{g}^{k}(\mathfrak{m})\,,\ k \geq 0\,,\ [X, \mathfrak{g}^{-1}(\mathfrak{m})] = \{0\}\,,\ \text{then } X = 0\,.$$

In [17], N. Tanaka proved that the full prolongation $\mathfrak{g}(\mathfrak{m})$ always exists and it is unique up to isomorphisms. Moreover, it can be defined inductively by

$$\mathfrak{g}^{i}(\mathfrak{m}) = \begin{cases} \mathfrak{m}^{i} & \text{if } i < 0\,,\\ \{A \in \mathrm{Der}(\mathfrak{m}, \mathfrak{m}) \ :\ A(\mathfrak{m}^{j}) \subset \mathfrak{m}^{j}\,, \forall j < 0\} & \text{if } i = 0\,,\\ \{A \in \mathrm{Der}(\mathfrak{m}, \sum_{h<i} \mathfrak{g}^{h}(\mathfrak{m})) \ :\ A(\mathfrak{m}^{j}) \subset \mathfrak{g}(\mathfrak{m})^{i+j}\,, \forall j < 0\} & \text{if } i > 0\,, \end{cases}$$

where $\mathrm{Der}(\mathfrak{m}, V)$ denotes the space of derivations of the Lie algebra $\mathfrak{m}$ with values in the $\mathfrak{m}$-module V.

Note that

$$\mathfrak{g}^{i}(\mathfrak{m}) \ = \ \Big\{ A \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}, \sum_{h<i} \mathfrak{g}^{h}(\mathfrak{m})) \,\Big|\, A(\mathfrak{g}^{h}(\mathfrak{m})) \subset \mathfrak{g}^{h+i}(\mathfrak{m}) \ \forall h < 0\,, \text{ and } [A(Y), Z] + [Y, A(Z)] = A([Y, Z]) \ \forall Y, Z \in \mathfrak{m} \Big\}\,. \tag{4}$$

3.2. **Prolongations of non-positively graded Lie algebras.** Consider now a non-positively graded Lie algebra $\mathfrak{m}+\mathfrak{g}^0 = \mathfrak{m}^{-d}+\cdots+\mathfrak{m}^{-1}+\mathfrak{g}^0$. The *full prolongation* of $\mathfrak{m}+\mathfrak{g}^0$ is the subalgebra

$$(\mathfrak{m}+\mathfrak{g}^0)^\infty = \mathfrak{m}^{-d}+\cdots+\mathfrak{m}^{-1}+\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2+\cdots$$

of $\mathfrak{g}(\mathfrak{m})$, defined inductively by

$$\mathfrak{g}^i = \{X \in \mathfrak{g}(\mathfrak{m})^i \,:\, [X,\mathfrak{m}^{-1}] \subset \mathfrak{g}^{i-1}\}\,, \ \text{ for any } i \geq 1\,.$$

Definition 3.1. *A graded Lie algebra* $\mathfrak{m}+\mathfrak{g}^0$ *is called of* finite type *if its full prolongation* $\mathfrak{g} = (\mathfrak{m}+\mathfrak{g}^0)^\infty$ *is a finite dimensional Lie algebra and it is called of* semisimple type *if* $\mathfrak{g}$ *is a finite dimensional semisimple Lie algebra.*

We have the following criterion (see [18], [3])

Lemma 3.2. *Let* $(\mathfrak{m} = \sum_{i<0}\mathfrak{m}^i, K_o)$ *be an integrable para-CR algebra and* $\mathfrak{g}^0$ *the subalgebras of* $\mathfrak{g}^0(\mathfrak{m})$ *consisting of any* $A \in \mathfrak{g}^0(\mathfrak{m})$ *such that* $A|_{\mathfrak{m}^{-1}}$ *commutes with* K_o*. Then the graded Lie algebra* $(\mathfrak{m}+\mathfrak{g}^0)$ *is of finite type if and only if* $\mathfrak{m}$ *is non-degenerate.*

The following result will be used in the last section (see e.g. [14], Theorem 3.21)

Lemma 3.3. *Let* $\mathfrak{g} = \sum_i \mathfrak{g}_i$ *be a fundamental effective semisimple graded Lie algebra such that* $\mathfrak{m}+\mathfrak{g}^0$ *is of finite type. Then* $\mathfrak{g}$ *coincides with the full prolongation* $(\mathfrak{m}+\mathfrak{g}^0)^\infty$ *of* $\mathfrak{m}+\mathfrak{g}^0$.

4. Standard almost para-CR manifolds

4.1. **Maximally homogeneous Tanaka structures.** A regular para-CR structure of type $(\mathfrak{m}, K_0)$ is of *finite type* or, respectively, of *semisimple type*, if the Lie algebra $(\mathfrak{m}+\mathfrak{g}^0)^\infty$ is finite-dimensional or, respectively, semisimple. Recall that $\mathfrak{g}^0 = Der(\mathfrak{m}, K_0)$ is the Lie algebra of Lie group $\mathrm{Aut}(\mathfrak{m}, K_0)$.
We recall the following (see [3])

Definition 4.1. *Let* $\mathfrak{m} = \mathfrak{m}^{-d}+\cdots+\mathfrak{m}^{-1}$ *be a negatively graded Lie algebra generated by* $\mathfrak{m}^{-1}$ *and* G^0 *a closed Lie subgroup of (grading preserving) automorphisms of* $\mathfrak{m}$*. A* Tanaka structure *of* type $(\mathfrak{m}, G^0)$ *on a manifold* M *is a regular distribution* $\mathcal{H} \subset TM$ *of type* $\mathfrak{m}$ *together with a principal* G^0*-bundle* $\pi : Q \to M$ *of adapted coframes of* $\mathcal{H}$*. A coframe* $\varphi : \mathcal{H}_x \to \mathfrak{m}^{-1}$ *is called* adapted *if it can be extended to an isomorphism* $\varphi : \mathfrak{m}_x \to \mathfrak{m}$ *of Lie algebra.*

We say that the Tanaka structure of type $(\mathfrak{m}, G^0)$ is of *finite type* (respectively *semisimple type* $(\mathfrak{m}, G^0)$), if the graded Lie algebra $\mathfrak{m}+\mathfrak{g}^0$ is of finite type (respectively semisimple type). Let P be a Lie subgroup of a connected Lie group G and $\mathfrak{p}$ (respectively $\mathfrak{g}$) the Lie algebra of P (respectively G).

Theorem 4.2. *Let* $(\pi : Q \to M, \mathcal{H})$ *be a Tanaka structure on* M *of semisimple type* $(\mathfrak{m}, G^0)$*. Then the Tanaka prolongation of* $(\pi, \mathcal{H})$ *is a* P*-principal*

bundle $\mathcal{G} \to M$, with the parabolic structure group P, equipped with a Cartan connection $\kappa : T\mathcal{G} \to \mathfrak{g}$, where $\mathfrak{g}$ is the full prolongation of $\mathfrak{m} + \mathfrak{g}^0$ and $\mathrm{Lie}P = \mathfrak{p} = \sum_{i\geq 0} \mathfrak{g}_i$. *Moreover,* $\mathrm{Aut}(\mathcal{H}, \pi)$ *is a Lie group and*

$$\dim \mathrm{Aut}(\mathcal{H}, \pi) \leq \dim \mathfrak{g}\,.$$

Let $(\mathcal{H}, \pi : Q \to M)$ be a Tanaka structure of semisimple type $(\mathfrak{m}, G^0)$ and $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty = \mathfrak{m} + \mathfrak{p}$ be the full prolongation of the non-positively graded Lie algebra $\mathfrak{m} + \mathfrak{g}^0$.

Definition 4.3. *A semisimple Tanaka structure $(\mathcal{H}, \pi : Q \to M)$ is called* maximally homogeneous *if the dimension of its automorphism group* $\mathrm{Aut}(\mathcal{H}, \pi)$ *is equal to* $\dim \mathfrak{g}$.

4.2. **Tanaka structures of semisimple type.** We construct maximally homogeneous Tanaka structures of semisimple type $(\mathfrak{m}, G^0)$ as follows. Let $G = \mathrm{Aut}(\mathfrak{g})$ be the Lie group of automorphisms of the graded Lie algebra $\mathfrak{g}$. Recall that G^0 is a closed subgroup of the automorphism group of the graded Lie algebra $\mathfrak{g}^- = \mathfrak{m}$. Since the Lie algebra $\mathfrak{g}$ is canonically associated with $\mathfrak{m}$, we can canonically extend the action of G^0 on $\mathfrak{m}$ to the action of G^0 on $\mathfrak{g}$ by automorphisms. In other words, we have an embedding $G^0 \hookrightarrow \mathrm{Aut}(\mathfrak{g}) = G$ as a closed subgroup. We denote by G^+ the connected (closed) subgroup of G with Lie algebra $\mathfrak{g}_+ = \sum_{p>0} \mathfrak{g}^p$. Then $P = G^0 \cdot G^+ \subset G$ is a (closed) parabolic subgroup of G. Let G/P be the corresponding flag manifold. We have a decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{p}$ and we identify $\mathfrak{m}$ with the tangent space $T_o(G/P)$. Then the natural action of G^0 on $\mathfrak{m}$ is the isotropy representation of G^0. We have a natural Tanaka structure $(\mathcal{H}, \pi : Q \to G/P)$ of type $(\mathfrak{m}, G^0)$, where $\mathcal{H}$ is the G-invariant distribution defined by $\mathfrak{m}^{-1}$ and Q is the G^0-bundle of coframes on $\mathcal{H}$.
Hence, the flag manifold G/P carries a natural maximally homogeneous Tanaka structure $(\mathcal{H}, \pi : Q \to G/P)$.
The universal covering F of the manifold G/P also has the induced Tanaka structure $(H_F, \pi_F : Q_F \to F)$ of type $(\mathfrak{m}, G^0)$ and the simply connected (connected) Lie group $\tilde{G}$ with the Lie algebra $\mathfrak{g}$ acts transitively and almost effectively on F as a group of automorphisms of this Tanaka structure. Moreover, the stabilizer in $\tilde{G}$ of an appropriate point $o \in F$ is the (connected) parabolic subgroup $\tilde{P}$ generated by the subalgebra $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^d$. The Tanaka structure $(\mathcal{H}, \pi : Q \to F = \tilde{G}/\tilde{P})$ is obviously maximally homogeneous and it is called the *standard (simply connected maximally homogeneous) Tanaka structure* of type $(\mathfrak{m}, G^0)$. We can state the following (see e.g. [2, Theor. 4.8])

Theorem 4.4. *Any maximally homogeneous Tanaka structure of semisimple type* $(\mathfrak{m}, G_0)$ *is isomorphic to the standard Tanaka structure on the simply connected flag manifold $F = \tilde{G}/\tilde{P}$ where $\tilde{G}$ is the simply connected semisimple Lie group with the Lie algebra $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty$ and $\tilde{P}$ is the parabolic subgroup generated by the subalgebra* $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^d$.

Let (HM, K) be a regular almost para-CR structure of type $(\mathfrak{m}, K_0)$. Assume that it has finite type, i.e. $m = \dim(\mathfrak{m} + \mathfrak{g}^0)^\infty < \infty$. According to the above definition, (HM, K) is *maximally homogeneous*, if it admits a (transitive) Lie group of automorphisms of dimension m.
By Theorem 4.4, a maximally homogeneous almost para-CR structure of semisimple type is locally equivalent to the standard structure associated with a gradation of a semisimple Lie algebra. In the following subsection we describe this correspondence in more details.

4.3. **Models of almost para-CR manifolds.** Let $\mathfrak{g} = \sum_{-d}^{d} \mathfrak{g}^i = \mathfrak{g}^- + \mathfrak{g}^0 + \mathfrak{g}^+$ be an effective fundamental gradation of a semisimple Lie algebra $\mathfrak{g}$ with negative part $\mathfrak{m} = \mathfrak{g}^- = \sum_{i<0} \mathfrak{g}^i$ and positive part $\mathfrak{g}^+ = \sum_{i>0} \mathfrak{g}^i$.
Denote by $F = \tilde{G}/\tilde{P}$ the simply connected real flag manifold associated with the graded Lie algebra $\mathfrak{g}$ where $\tilde{G}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and $\tilde{P} = G^0 G^+$ is the connected subgroup generated by the Lie subalgebra $\mathfrak{g}^0 + \mathfrak{g}^+$.
We will identify the tangent space T_oF at the point $o = eP$ with the subspace

$$\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{m}\,.$$

Since the subspace $(\mathfrak{g}^{-1} + \mathfrak{p})/\mathfrak{p} \subset T_oF$ is invariant under the isotropy representation of P, it defines an invariant distribution $\mathcal{H}$ on F. Since the gradation is fundamental, one can easily check that, for any $x \in F$, the negatively graded Lie algebra $\mathfrak{m}(x)$ associated with $\mathcal{H}$ is isomorphic to the Lie algebra $\mathfrak{m}$. Moreover, let

$$\mathfrak{g}^{-1} = \mathfrak{g}^{-1}_+ + \mathfrak{g}^{-1}_- \tag{5}$$

be a decomposition of the G^0-module $\mathfrak{g}^{-1}$ into a sum of two submodules and K_0 the associated $\mathrm{ad}_{\mathfrak{g}_0}$-invariant endomorphism such that $\mathfrak{g}^{-1}_\pm$ are ± 1-eigenspaces of K_0.
The decomposition (5) defines two invariant complementary subdistributions $\mathcal{H}_\pm$ of the distribution $\mathcal{H} \subset TF$ associated with $\mathfrak{g}^{-1}$ and K_0 defines $\tilde{G}$-invariant para-CR structure (HF, K) on F. It is the standard para-CR structure associated with the graded Lie algebra $\mathfrak{g}$ and the decomposition (5). We get the following theorem (see also [2, Theor. 5.1])

Theorem 4.5. *Let $F = \tilde{G}/\tilde{P}$ be the simply connected flag manifold associated with a (real) semisimple effective fundamental graded Lie algebra $\mathfrak{g}$. A decomposition $\mathfrak{g}^{-1} = \mathfrak{g}^{-1}_+ + \mathfrak{g}^{-1}_-$ of $\mathfrak{g}^{-1}$ into complementary G^0-submodules $\mathfrak{g}^{-1}_\pm$ determines an invariant almost para-CR structure (HM, K) such that ± 1-eigenspaces $H_\pm M$ of K are subdistributions of HM associated with $\mathfrak{g}^{-1}_\pm$. Conversely, any standard almost para-CR structure (HM, K) on F can be obtained in such a way.*
Moreover, (HM, K) is:

(1) *an almost para-CR structure if $\mathfrak{g}^{-1}_+$ and $\mathfrak{g}^{-1}_-$ have the same dimensions,*

(2) *a para-CR structure if and only if* $\mathfrak{g}_+^{-1}$ *and* $\mathfrak{g}_-^{-1}$ *are commutative subalgebras of* $\mathfrak{g}$*,*
(3) *non-degenerate if and only if* $\mathfrak{g}$ *has no graded ideals of depth one.*

By Theorem 4.5, the classification of maximally homogeneous para-CR structures of semisimple type, up to local isomorphisms (i.e. up to coverings), reduces to the description of all gradation of semisimple Lie algebras $\mathfrak{g}$ and to decomposition of the $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$ into irreducible submodules. We will give such a description for complex and real semisimple Lie algebras in the next two sections.

5. Fundamental gradations of a complex semisimple Lie algebra

We recall here the construction of a gradation of a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

be the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We denote by

$$\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset R$$

a system of simple roots of the root system R and associate to each simple root α_i (or corresponding vertex of the Dynkin diagram) a non-negative integer d_i. Using the *label vector* $\vec{d} = (d_1, \dots, d_\ell)$, we define the *degree* of a root $\alpha = \sum_{i=1}^{\ell} k_i \alpha_i$ by

$$d(\alpha) = \sum_{i=1}^{\ell} k_i d_i \, .$$

This defines a gradation of $\mathfrak{g}$ by the conditions

$$d(\mathfrak{h}) = 0 \, , \qquad d(\mathfrak{g}_\alpha) = d(\alpha), \quad \forall \alpha \in R \, ,$$

which is called the *gradation associated with the label vector* $\vec{d}$.
We denote by $d \in \mathfrak{h}$ the corresponding grading element. Then $d(\alpha) = \alpha(d)$. Any gradation of a complex semisimple Lie algebra $\mathfrak{g}$ is conjugated to a gradation of such a type (see [11]). In particular, it has the form

$$\mathfrak{g} = \mathfrak{g}^{-k} + \cdots + \mathfrak{g}^0 + \cdots + \mathfrak{g}^k \, ,$$

where $\mathfrak{g}^0$ is a reductive subalgebra of $\mathfrak{g}$ and the grading spaces $\mathfrak{g}^{-i}$ and $\mathfrak{g}^i$ are dual with respect to the Killing form. It is clear now that any graded semisimple Lie algebra is a direct sum of graded simple Lie algebras. Hence, it is sufficient to describe gradations of simple Lie algebras.
We need the following (see [19])

Lemma 5.1. *The gradation of a complex semisimple Lie algebra* $\mathfrak{g}$ *associated with a label vector* $\vec{d} = (d_1, \dots, d_\ell)$ *is fundamental if and only if all labels* $d_i \in \{0, 1\}$*.*

Let $\Pi^1 \subset \Pi$ be a set of simple roots. We denote by $\vec{d}_{\Pi^1}$ the label vector which associates label one to the roots in Π^1 and label zero to the other simple roots.
Now we describe the depth of a fundamental gradation.
Let μ be the maximal root with respect to the fundamental system Π. It can be written as a linear combination

$$\mu = m_1\alpha_1 + \cdots + m_\ell\alpha_\ell \tag{6}$$

of fundamental roots, where the coefficient m_i is a positive integer called the *Dynkin mark associated* with α_i.

Lemma 5.2. *Let* $\Pi^1 = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\} \subset \Pi$ *be a set of simple roots. Then the depth* k *of the fundamental gradation defined by the label vector* $\vec{d}_{\Pi^1}$ *is given by*

$$k = m_{i_1} + m_{i_2} + \cdots + m_{i_s}\,.$$

Proof. The depth k of the gradation is equal to the maximal degree $d(\alpha)$, α being a root. If $\alpha = k_1\alpha_1 + \cdots + k_\ell\alpha_\ell$ is the decomposition of a root α with respect to simple roots, then

$$d(\alpha) = k_{i_1} + \cdots + k_{i_s} \leq d(\mu) = m_{i_1} + \cdots + m_{i_s} = k\,.$$

□

Irreducible submodules of the $\mathfrak{g}^0$-module $\mathfrak{g}^1$. Let $\mathfrak{g} = \sum \mathfrak{g}^i$ be a fundamental gradation of a complex semisimple Lie algebra $\mathfrak{g}$, defined by a label vector $\vec{d}$. Following [11], we describe the decomposition of a $\mathfrak{g}^0$-module into irreducible submodules. Set

$$R^i = \{\alpha \in R \mid d(\alpha) = i\} = \{\alpha \in R \mid \mathfrak{g}_\alpha \subset \mathfrak{g}^i\}$$

and

$$\Pi^i = \Pi \cap R^i = \{\alpha \in \Pi \mid d(\alpha) = i\}\,.$$

For any simple root $\gamma \in \Pi$, we put

$$R(\gamma) = \{\gamma + (R^0 \cup \{0\})\} \cap R = \{\alpha = \gamma + \phi^0 \in R,\;\; \phi^0 \in R^0 \cup \{0\}\}.$$

We associate to any set of roots $Q \subset R$ a subspace

$$\mathfrak{g}(Q) = \sum_{\alpha \in Q} \mathfrak{g}_\alpha \subset \mathfrak{g}\,.$$

Proposition 5.3. ([11]) *The decomposition of a* $\mathfrak{g}^0$*-module* $\mathfrak{g}^1$ *into irreducible submodules is given by*

$$\mathfrak{g}^1 = \sum_{\gamma \in \Pi^1} \mathfrak{g}(R(\gamma))\,.$$

Moreover, γ *is a lowest weight of the irreducible submodule* $\mathfrak{g}(R(\gamma))$*. In particular, the number of the irreducible components is equal to the number* $\#\Pi^1$ *of the simple roots of degree* 1*.*

Since the $\mathfrak{g}^0$-modules $\mathfrak{g}^i$ and $\mathfrak{g}^{-i}$ are dual, Proposition 5.3 gives also the decomposition of the $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$ into irreducible submodules.

6. Fundamental gradations of a real semisimple Lie algebra

6.1. **Real forms of a complex semisimple Lie algebra.** Now we recall the description of a real form of a complex semisimple Lie algebra in terms of Satake diagrams. It is sufficient to do this for complex simple Lie algebras.

Any real form of a complex semisimple Lie algebra $\mathfrak{g}$ is the fixed points set $\mathfrak{g}^\sigma$ of an antilinear involution σ, that is, an antilinear map $\sigma : \mathfrak{g} \to \mathfrak{g}$, which is an automorphism of $\mathfrak{g}$ as a real algebra, such that $\sigma^2 = \mathrm{id}$. We fix a Cartan decomposition

$$\mathfrak{g}^\sigma = \mathfrak{k} + \mathfrak{m}$$

of the real form $\mathfrak{g}^\sigma$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}^\sigma$ and $\mathfrak{m}$ is its orthogonal complement with respect to the Killing form B. Let

$$\mathfrak{h}^\sigma = \mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{m}$$

be a Cartan subalgebra of $\mathfrak{g}^\sigma$ which is consistent with this decomposition and such that $\mathfrak{h}_\mathfrak{m} = \mathfrak{h} \cap \mathfrak{m}$ has maximal dimension. Then the root decomposition of $\mathfrak{g}^\sigma$, with respect to the subalgebra $\mathfrak{h}^\sigma$, can be written as

$$\mathfrak{g}^\sigma = \mathfrak{h}^\sigma + \sum_{\lambda \in \Sigma} \mathfrak{g}^\sigma_\lambda \,,$$

where $\Sigma \subset (\mathfrak{h}^\sigma)^*$ is a (non-reduced) root system. The number $m_\lambda = \dim \mathfrak{g}_\lambda$ is the *multiplicity* of a root $\lambda \in \Sigma$.

Denote by $\mathfrak{h} = (\mathfrak{h}^\sigma)^\mathbb{C}$ the complexification of $\mathfrak{h}^\sigma$ which is a σ-invariant Cartan subalgebra. We denote by σ^* the induced antilinear action of σ on $\mathfrak{h}^*$ given by

$$\sigma^* \alpha = \overline{\alpha \circ \sigma} \,, \quad \alpha \in \mathfrak{h}^* \,.$$

Consider the root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

of the Lie algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. Note that σ^* preserves the root system R, i.e. $\sigma^* R = R$. Now we relate the root space decomposition of $\mathfrak{g}^\sigma$ and $\mathfrak{g}$. We define the subsystem of compact roots $R_\bullet$ by

$$R_\bullet = \{\alpha \in R \mid \sigma^*\alpha = -\alpha\} = \{\alpha \mid \alpha(\mathfrak{h}_\mathfrak{m}) = 0\}$$

and denote by $R' = R \setminus R_\bullet$ the complementary set of non-compact roots. We can choose a system Π of simple roots of R such that the corresponding system of positive roots R_+ satisfies the condition: $R'_+ = R' \cap R_+$ is σ-invariant. In this case, Π is called a *σ-fundamental system* of roots. We denote by $\Pi_\bullet = \Pi \cap R_\bullet$ the set of compact simple roots (which are also called black) and by $\Pi' = \Pi \setminus \Pi_\bullet$ the non-compact simple roots (called white). The action of σ^* on white roots satisfies the following property:

for any $\alpha \in \Pi'$ there exists a unique $\alpha' \in \Pi'$ such that $\sigma^*\alpha - \alpha'$ is a linear combination of black roots, i.e.

$$\sigma^*\alpha = \alpha' + \sum_{\beta \in \Pi_\bullet} k_\beta \beta, \quad k_\beta \in \mathbb{N}\,.$$

In this case, we say that the roots α, α' are *σ-equivalent* and we will write $\alpha \sim \alpha'$. The information about fundamental system ($\Pi = \Pi_\bullet \cup \Pi'$) together with the σ-equivalence can be visualized in terms of the *Satake diagram*, which is defined as follows.
On the Dynkin diagram of the system of simple roots Π, we paint the vertices which correspond to black roots into black and we join the vertices which correspond to σ-equivalent roots α, α' by a curved arrow.
By a slight abuse of notation, we will refer to the σ-fundamental system $\Pi = \Pi_\bullet \cup \Pi'$, together with the σ-equivalence $\sim$, as the *Satake diagram.* This diagram is determined by the real form $\mathfrak{g}^\sigma$ of a complex simple Lie algebra $\mathfrak{g}$ and does not depend on the choice of a Cartan subalgebra and a σ-fundamental system. The list of Satake diagram of real simple Lie algebras is known (see e.g. [11]).
Conversely, Satake diagram ($\Pi = \Pi_\bullet \cup \Pi', \sim$) allows to reconstruct the action of σ^* on Π, hence on $\mathfrak{h}^*$. This action can be canonically extended to the antilinear involution σ of the complex Lie algebra $\mathfrak{g}$. Hence, *there is a natural $1-1$ correspondence between Satake diagrams subordinated to the Dynkin diagram of a complex semisimple Lie algebra $\mathfrak{g}$, up to isomorphisms, and real forms $\mathfrak{g}^\sigma$ of $\mathfrak{g}$, up to conjugations.*

6.2. **Gradations of a real semisimple Lie algebra.** Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{g}^\sigma$ be a real form of $\mathfrak{g}$ with a Satake diagram ($\Pi = \Pi_\bullet \cup \Pi', \sim$). Let $\vec{d} = (d_1, \dots, d_\ell)$ be a label vector of the simple roots system Π and $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}^i$ be the corresponding gradation of $\mathfrak{g}$, with the grading element $d \in \mathfrak{h} \subset \mathfrak{g}$.
The following theorem gives necessary and sufficient conditions in order that this gradation induces a gradation

$$\mathfrak{g}^\sigma = \sum_{i \in \mathbb{Z}} \mathfrak{g}^\sigma \cap \mathfrak{g}^i$$

of the real form $\mathfrak{g}^\sigma$. This means that the grading element d belongs to $\mathfrak{g}^\sigma$. We denote by $\Pi^0 \subset \Pi$ the set of simple roots with label zero.

Theorem 6.1. ([10]) *A gradation of a complex semisimple Lie algebra $\mathfrak{g}$, associated with a label vector $\vec{d} = (d_1, \dots, d_\ell)$, induces a gradation of the real form $\mathfrak{g}^\sigma$, which corresponds to a Satake diagram ($\Pi = \Pi_\bullet \cup \Pi', \sim$) if and only if the following two conditions hold:*

i) $\Pi_\bullet \subset \Pi^0$, *i.e. any black vertex of the Satake diagram has label zero;*

ii) *if $\alpha \sim \alpha'$ for $\alpha,\ \alpha' \in \Pi \setminus \Pi_\bullet$, then $d(\alpha) = d(\alpha')$, i.e. white vertices of the Satake diagram which are joint by a curved arrow have the same label.*

A label vector $\vec{d} = (d_1, \dots, d_\ell)$ of a Satake diagram $(\Pi = \{\alpha_1, \dots, \alpha_\ell\} = \Pi_\bullet \cup \Pi', \sim)$ and the corresponding gradation of $\mathfrak{g}$ are called of *real type* if they satisfy conditions i) and ii) of the theorem above, that is black vertices have label zero and vertices related by a curved arrow have the same label. Hence, we can state Theorem 6.1 as follows

Corollary 6.2. *There exists a natural $1-1$ correspondence between label vectors $\vec{d}$ of real type of a Satake diagram of a real semisimple Lie algebra $\mathfrak{g}^\sigma$ and gradations of $\mathfrak{g}^\sigma$. The gradation of $\mathfrak{g}^\sigma$ is fundamental if and only if the corresponding gradation of $\mathfrak{g}$ is fundamental, i.e. $\vec{d} = \vec{d_{\Pi^1}}$.*

Irreducible submodules of the $\mathfrak{g}^0$-module $\mathfrak{g}^1$. Let $\mathfrak{g} = \sum \mathfrak{g}^i$ be a gradation of a complex semisimple Lie algebra $\mathfrak{g}$ with grading element d and $\mathfrak{g}^\sigma = \sum (\mathfrak{g}^\sigma)^i = \sum \mathfrak{g}^i \cap \mathfrak{g}^\sigma$ be a real form of $\mathfrak{g}$, consistent with this gradation. We denote by $(\Pi = \Pi_\bullet \cup \Pi',\ \sim)$ the Satake diagram of $\mathfrak{g}^\sigma$.
By Proposition 5.3, the decomposition of $\mathfrak{g}^1$ into irreducible $\mathfrak{g}^0$-submodules is given by $\mathfrak{g}^1 = \sum_{\gamma \in \Pi^1} \mathfrak{g}(R(\gamma))$, where Π^1 is the set of simple roots of label one. The following obvious proposition describes the decomposition of $(\mathfrak{g}^\sigma)^0$-module $(\mathfrak{g}^\sigma)^1$ into irreducible submodules.

Proposition 6.3. *For any simple root $\gamma \in \Pi^1$ of label one, there are two possibilities:*

i) *$\sigma^*\gamma = \gamma + \sum_{\beta \in \Pi_\bullet} k_\beta \beta$. Then $\sigma^*\gamma \in R(\gamma)$ and the $\mathfrak{g}^0$-module $\mathfrak{g}(R(\gamma))$ is σ-invariant;*
ii) *$\sigma^*\gamma = \gamma' + \sum_{\beta \in \Pi_\bullet} k_\beta \beta$, where $\gamma \neq \gamma' \in \Pi^1$. Then, $\sigma^* R(\gamma) = R(\gamma')$ and the two irreducible $\mathfrak{g}^0$-modules $\mathfrak{g}(R(\gamma))$ and $\mathfrak{g}(R(\gamma'))$ determine one irreducible submodule $\mathfrak{g}^\sigma \cap (\mathfrak{g}(R(\gamma)) + \mathfrak{g}(R(\gamma')))$ of $\mathfrak{g}^\sigma$.*

Corollary 6.4. *Let $\mathfrak{g}^\sigma = \sum (\mathfrak{g}^\sigma)^i$ be the gradation of a real semisimple Lie algebra $\mathfrak{g}^\sigma$, associated with a label vector $\vec{d}$ of real type. Then irreducible submodules of the $(\mathfrak{g}^\sigma)^0$-module $(\mathfrak{g}^\sigma)^{-1}$ correspond to vertices γ with label one without curved arrow and to pairs (γ, γ') of vertices with label one related by a curved arrow. In particular, a decomposition of the $(\mathfrak{g}^\sigma)^0$-module $(\mathfrak{g}^\sigma)^{-1}$ is determined by a decomposition of the set Π^1 of vertices with label 1 into a disjoint union $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ such that equivalent vertices belong to the same component. The corresponding submodules $(\mathfrak{g}^\sigma)^{-1}_+$ and $(\mathfrak{g}^\sigma)^1_-$ are given by*

$$(\mathfrak{g}^\sigma)^{-1}_\pm = \mathfrak{g}^\sigma \cap \sum_{\gamma \in \Pi^1_\pm} \mathfrak{g}(R(-\gamma)). \tag{7}$$

We will always assume that a decomposition of Π^1 satisfies the above property.

7. Classification of Maximally homogeneous para-CR manifolds

Let $\mathfrak{g}^\sigma$ be a real semisimple Lie algebra associated with a Satake diagram $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ with the fundamental gradation defined by a subset $\Pi^1 \subset \Pi'$ and $F = \tilde{G}/\tilde{P}$ be the associated flag manifold.

By Theorem 4.5, an almost para-CR structure on $F = \tilde{G}/\tilde{P}$ associated with a decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ is integrable (i.e. a para-CR structure) if and only if the $(\mathfrak{g}^\sigma)^0$-submodules $(\mathfrak{g}^\sigma)^{-1}_+$ and $(\mathfrak{g}^\sigma)^{-1}_-$ given by (7) are Abelian subalgebras of $\mathfrak{g}^\sigma$. In order to give an integrability criterion, we introduce the following definitions.

Definition 7.1. *Let R be a system of roots and Π be a system of simple roots. A subset $\Pi^1 \subset \Pi$ is said to be* admissible *if Π^1 contains at least two roots and there are no roots of R of the form*

$$2\alpha + \sum k_i \phi_i \,, \quad \text{with } \alpha \in \Pi^1 \,,\ \phi_i \in \Pi_0 = \Pi \setminus \Pi^1. \tag{8}$$

Definition 7.2. *Let $\mathfrak{g}^\sigma$ be a real semisimple Lie algebra with a fundamental gradation defined by a subset $\Pi^1 \subset \Pi'$. We say that a decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ is* alternate *if the following conditions hold:*

i) *if $\alpha \in \Pi^1_\pm$ and $\alpha' \sim \alpha$, then $\alpha' \in \Pi^1_\pm$;*

ii) *the vertices in Π^1_+ and Π^1_- appear in the Satake diagram in alternate order. This means that each connected component of the graph obtained deleting vertices in Π^1_+ (respectively in Π^1_-) has not more than one vertex in Π^1_- (respectively in Π^1_+).*

We are ready to state the following

Proposition 7.3. *Let $\mathfrak{g}^\sigma$ be a semisimple real Lie algebra with the fundamental gradation associated with a subset $\Pi^1 \subset \Pi$ and $F = \tilde{G}/\tilde{P}$ the associated flag manifold. A decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ defines a para-CR structure on the flag manifold F if and only if the subset Π^1 is admissible and the decomposition of Π^1 is alternate.*

For the proof we need the following lemma.

Lemma 7.4. *The subspace $\mathfrak{g}^1_+ = \sum_{\gamma \in \Pi^1_+} \mathfrak{g}(R(\gamma))$ (hence also the subspace $(\mathfrak{g}^\sigma)^1_+ = \mathfrak{g}^\sigma \cap \mathfrak{g}^1_+$) which corresponds to a subset $\Pi^1_+ \subset \Pi^1$ is an Abelian subalgebra if and only if there is no root β of the form*

$$\beta = \alpha + \alpha' + \sum k_i \phi_i \tag{9}$$

where $\alpha, \alpha' \in \Pi^1_+$ and $\phi_i \in \Pi^0$. The case $\alpha = \alpha'$ is allowed.

Proof. If such a root β exists, then $[\mathfrak{g}(R(\alpha), \mathfrak{g}(R(\alpha'))] \neq 0$ and $\mathfrak{g}^1_+$ is not an Abelian subalgebra. The converse is also clear. □

Proof of Proposition 7.3. Let $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ be a decomposition of Π^1. The condition (9) for $\alpha = \alpha'$ is fulfilled if and only if Π^1 is admissible. Assume now that two different vertices α, α' in Π^1_+ are connected in the Satake

diagram by vertices in $\Pi^0 = \Pi \setminus \Pi^1$. Then there is a root of the form (9) and $\mathfrak{g}^1_+$ is not a commutative subalgebra. This shows that the decomposition which defines a para-CR structure on F must be alternate.
Conversely, assume that the decomposition is alternate. Then any two vertices $\alpha, \alpha' \in \Pi^1_+$ belong to different connected components of the Satake graph with deleting Π^1_-. This implies that there is no root of the form (9) for $\alpha \neq \alpha'$. Then Lemma 7.4 shows that $(\mathfrak{g}^\sigma)^1_+$ is a commutative subalgebra. The same argument is applied also for $(\mathfrak{g}^\sigma)^1_-$. □

We enumerate simple roots of complex simple Lie $\mathfrak{g}$ algebras as in [4]. Let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be the simple roots of $\mathfrak{g}$, which are identified with vertices of the corresponding Dynkin diagram. We denote the elements of a subset $\Pi^1 \subset \Pi$ (respectively $\Pi^1 \subset \Pi'$) which defines a fundamental gradation of $\mathfrak{g}$ (respectively $\mathfrak{g}^\sigma$) by

$$\alpha_{i_1}, \dots, \alpha_{i_k}, \quad i_1 < i_2 < \dots < i_k.$$

Proposition 7.5. *Let Π be a system of simple roots of a root system R of a complex simple Lie algebra $\mathfrak{g}$. Then a subset $\Pi^1 \subset \Pi$ of at least two elements is admissible (see Definition 7.1) in the following cases:*

- *for $\mathfrak{g} = A_\ell$, in all cases;*
- *for $\mathfrak{g} = B_\ell$, under the condition: $i_k = i_{k-1} + 1$;*
- *for $\mathfrak{g} = C_\ell$, under the condition: $i_k = \ell$;*
- *for $\mathfrak{g} = D_\ell$, under the condition: if $i_k < \ell - 1$, then $i_k = i_{k-1} + 1$;*
- *for $\mathfrak{g} = E_6$, in all cases except the following ones:*

$$\{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_3, \alpha_6\}, \{\alpha_4, \alpha_6\}, \{\alpha_1, \alpha_4, \alpha_6\};$$

- *for $\mathfrak{g} = E_7$, in all cases except the following ones:*

$$\begin{gathered}\{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_3, \alpha_6\}, \{\alpha_4, \alpha_6\}, \{\alpha_1, \alpha_6\}, \\ \{\alpha_2, \alpha_7\}, \{\alpha_3, \alpha_7\}, \{\alpha_4, \alpha_7\}, \{\alpha_5, \alpha_7\}, \\ \{\alpha_1, \alpha_4, \alpha_6\}, \{\alpha_1, \alpha_4, \alpha_7\}, \{\alpha_1, \alpha_5, \alpha_7\}, \{\alpha_3, \alpha_6, \alpha_7\}, \{\alpha_4, \alpha_6, \alpha_7\}, \\ \{\alpha_1, \alpha_4, \alpha_6, \alpha_7\};\end{gathered}$$

- *for $\mathfrak{g} = E_8$, in all cases except the following ones:*

$$\begin{gathered}\{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_3, \alpha_6\}, \{\alpha_4, \alpha_6\}, \{\alpha_1, \alpha_6\}, \\ \{\alpha_2, \alpha_7\}, \{\alpha_3, \alpha_7\}, \{\alpha_4, \alpha_7\}, \{\alpha_5, \alpha_7\}, \\ \{\alpha_1, \alpha_4, \alpha_6\}, \{\alpha_1, \alpha_4, \alpha_7\}, \{\alpha_1, \alpha_5, \alpha_7\}, \{\alpha_3, \alpha_6, \alpha_7\}, \{\alpha_4, \alpha_6, \alpha_7\}, \\ \{\alpha_1, \alpha_4, \alpha_6, \alpha_7\}, \\ \{\alpha_1, \alpha_7\}, \{\alpha_1, \alpha_8\}, \{\alpha_2, \alpha_8\}, \{\alpha_3, \alpha_8\}, \{\alpha_4, \alpha_8\}, \{\alpha_5, \alpha_8\}, \{\alpha_6, \alpha_8\}, \\ \{\alpha_1, \alpha_4, \alpha_8\}, \{\alpha_1, \alpha_5, \alpha_8\}, \{\alpha_3, \alpha_6, \alpha_8\}, \{\alpha_4, \alpha_6, \alpha_8\}, \{\alpha_1, \alpha_6, \alpha_8\}, \\ \{\alpha_2, \alpha_7, \alpha_8\}, \{\alpha_3, \alpha_7, \alpha_8\}, \{\alpha_4, \alpha_7, \alpha_8\}, \{\alpha_5, \alpha_7, \alpha_8\}, \\ \{\alpha_1, \alpha_4, \alpha_6, \alpha_8\}, \{\alpha_1, \alpha_4, \alpha_7, \alpha_8\}, \{\alpha_1, \alpha_5, \alpha_7, \alpha_8\}, \{\alpha_3, \alpha_6, \alpha_7, \alpha_8\}, \\ \{\alpha_4, \alpha_6, \alpha_7, \alpha_8\}, \\ \{\alpha_1, \alpha_4, \alpha_6, \alpha_7, \alpha_8\};\end{gathered}$$

- *for* $\mathfrak{g} = F_4$, *in all cases except the following ones:*

$$\{\alpha_1, \alpha_3\}\,,\ \{\alpha_1, \alpha_4\}\,,\ \{\alpha_2, \alpha_4\}\,,\ \{\alpha_3, \alpha_4\}\,,\ \{\alpha_1, \alpha_3, \alpha_4\}\,;$$

- *for* $\mathfrak{g} = G_2$, *in the case* $\{\alpha_1, \alpha_2\}$.

In cases different from D_ℓ, E_6, E_7 *and* E_8, *for any* Π^1 *given as above it is possible to give an alternate decomposition* $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$.
For D_ℓ, *an alternate decomposition of* Π^1 *can be given in the following cases:*

- $\alpha_{\ell-2} \in \Pi^1$,
- Π^1 *is contained in at most two of the branches issuing from* $\alpha_{\ell-2}$.

For E_6, E_7 *and* E_8, *an alternate decomposition of* Π^1 *can be given in the following cases:*

- $\alpha_4 \in \Pi^1$,
- Π^1 *is contained in at most two of the branches issuing from* α_4.

Proof. We have to describe all subsets Π^1 of Π which satisfy (8). This condition can be reformulated as follows. For any $\alpha \in \Pi^1$, denote by Π_α the connected component of the subdiagram of the Dynkin diagram Π obtained by deleting vertices in $\Pi^1 \setminus \{\alpha\}$ and containing α. Then the root system associated with Π_α has no roots of the form

$$\beta = 2\alpha + \sum_{\phi \in \Pi_\alpha \setminus \{\alpha\}} k_\phi \phi.$$

Using this condition and the decomposition of any root into a linear combination of simple roots, one can prove the proposition.
In the case of A_ℓ, any root has coefficient $0, 1$ in the decomposition into simple roots. Hence, any decomposition satisfies the property (8).
In the case of B_ℓ, any root which has coefficient 2 has the form

$$\sum_{i \le h < j} \alpha_h + 2 \sum_{j \le h \le \ell} \alpha_h\,, \qquad (1 \le i < j \le \ell)\,.$$

Hence the condition (8) holds if and only if the last two roots in Π^1 are consecutive, i.e. $i_{k-1} + 1 = i_k$.
In the case of C_ℓ, the roots with a coefficient 2 are given by

$$\sum_{i \le h < j} \alpha_h + 2 \sum_{j \le h < \ell} \alpha_h + \alpha_\ell\,, \qquad (1 \le i < j \le \ell)\,,$$
$$2 \sum_{i \le h < \ell} \alpha_h + \alpha_\ell\,, \qquad (1 \le i < \ell)\,.$$

The second formula implies that there are no roots of the form given in (8) if and only if $i_k = \ell$.
In the case of D_ℓ, the roots with a coefficient 2 are

$$\sum_{i \le h < j} \alpha_h + 2 \sum_{j \le h < \ell - 1} \alpha_h + \alpha_{\ell-1} + \alpha_\ell\,, \qquad (1 \le i < j < \ell - 1)\,.$$

The condition (8) fails if and only if the last two roots $\alpha_{i_{k-1}}, \alpha_{i_k}$ satisfy $i_{k-1} < i_k - 1$ and $i_k < \ell - 1$.

The case of exceptional Lie algebras can be treated in a similar way, by using tables in [4]. $\square$

Let $\Pi^1 \subset \Pi'$ be an admissible subset which defines a fundamental gradation of $\mathfrak{g}^\sigma$. An alternate decomposition of $\Pi^1 = \Pi^1_+ \cup \Pi^1_+$ can be given if the conditions of Proposition 7.5 are satisfied and, in addition, the following ones hold:

- for $\mathfrak{su}(p,q)$, it has to be $q = p$ and $\alpha_p \in \Pi^1$;
- for $\mathfrak{so}(\ell-1, \ell+1)$, it has to be $\Pi^1 \cap \{\alpha_{\ell-1}, \alpha_\ell\} = \emptyset$ or $\{\alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell\} \subset \Pi^1$;
- for E_6II, it has to be $\alpha_4 \in \Pi^1$ and if $\alpha_2 \notin \Pi^1$, then $\{\alpha_3, \alpha_5\} \subset \Pi^1$;

while for $\mathfrak{so}^*(2\ell)$ and E_6III there is no alternate decomposition of Π^1.

Proposition 7.3 implies the following final theorem.

Theorem 7.6. *Let $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ be a Satake diagram of a simple real Lie algebra $\mathfrak{g}^\sigma$ and $\Pi^1 \subset \Pi'$ be an admissible subset as described above. Let $\tilde{G}$ be the simply connected Lie group with the Lie algebra $\mathfrak{g}^\sigma$ and $\tilde{P}$ be the parabolic subgroup of $\tilde{G}$ generated by the non-negatively graded subalgebra*

$$\mathfrak{p} = \sum_{i \geq 0} (\mathfrak{g}^\sigma)^i$$

associated with the grading element $\vec{d}_{\Pi^1}$. Then the alternate decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ defines a decomposition

$$(\mathfrak{g}^\sigma)^1 = (\mathfrak{g}^\sigma)^1_+ + (\mathfrak{g}^\sigma)^1_-$$

of the $(\mathfrak{g}^\sigma)^0$-module $(\mathfrak{g}^\sigma)^1$ into a sum of two commutative subalgebras. This decomposition determines an invariant para-CR structure on the simply connected flag manifold $F = \tilde{G}/\tilde{P}$. Moreover, any simply connected maximally homogeneous para-CR manifolds of semisimple type is a direct product of such manifolds.

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